INITIAL-BOUNDARY VALUE PROBLEM FOR THE WAVE PROPAGATION EQUATION ON LADDER-SHAPED GRAPHS

Kenjaboyeva M. H.
Teacher of Chirchik State Pedagogical University
Chirchik, Uzbekistan.

Annotation: In this paper, we investigate local boundary value problems for the wave equation on ladder type metric graph. The main goal of this work is to study the uniqueness and the existence of solution of the formulated problem. Using by the method of separation of variables we find a solution of the investigated problem in the form of the Fourier series. Existence and uniqueness of solution of the considered problems are defined.

Keywords: Metric graph, ladder type metric graph, method of separation of variables, apriori estimates.

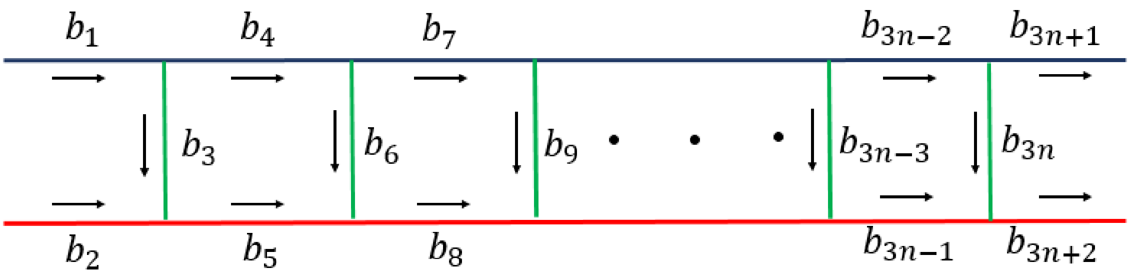
Introduction

Initial-boundary problems for differential equations on metric graphs is a relatively new field of study, and the linear Schrödinger equation (quantum graph) on graphs began to be studied mainly in the 1990s[1]. Other types of equations on graphs began to be studied mainly in the 2010s. We can cite the results obtained on the nonlinear Schrödinger equation[2], the Sine–Gordon equation, and the Korteweg–De Vries equations. G. Berkolaiko used them in his work to study various physical processes on three-sided star graphs[3]. ZA Sobirov and MR Eshimbetov studied heat dissipation processes on staircase graphs[4]. The initial-boundary problem for the wave propagation equation on staircase graphs is considered in this work for the first time.

The study of wave processes in ladder graphs is a practically important issue. Such graphs serve as a model for studying the propagation of impulses in DNA and RNA molecules.

1. The problem statement

The edges of us $3n+2$, $(n \in N)$ formed by joining together sections of equal length Γ Given a ladder graph (Figure 1), find the edges of the graph. b_k, $k=\overline{1,3n+2}$ We define it in the form of. Each b_k identifiable in the bond x_k we have the coordinates x and all b_k to the edges $(0,L)$ we adjust the cut, that is $b_k \sim (0,L)$, $k=\overline{1,3n+2}$.



(Drawing 1)

At each edge of the graph, we consider the following wave propagation equation

$$u_{k\,tt}(x,t)=a^2u_{k\,xx}(x,t)+f_k(x,t),\quad (x,t)\in((0,L)\times(0,T)),\quad k=\overline{1,3n+2}. \tag{1}$$

Here $u_k=(u_1,u_2,u_3,\dots,u_{3n+2})^T$, $f_k=(f_1,f_2,f_3,\dots,f_{3n+2})^T$. $f_k(x,t)$ given specific functions $k=\overline{1,3n+2}$.

Γ Let's look at the following problem for equation (1) on the graph.

Issue1. The wave propagation equation given in the form (1) $b_k\times(0,T)$ identified in the field and the following:

elementary

$$u_k(x,t)\big|_{t=0}=\varphi_k(x),\quad u_{k\,t}(x,t)\big|_{t=0}=\psi_k(x),\quad 0\le x\le L,\quad k=\overline{1,3n+2}, \tag{2}$$

homogeneous border

$$u_k(x,t)\big|_{x=0}=0,\ k\in\{1,2\},\quad u_k(x,t)\big|_{x=L}=0,\ k\in\{3n+1,3n+2\},\ t\ge 0, \tag{3}$$

and the following connection conditions (Kirchhoff's conditions)

$$u_{3k-2}(x,t)\big|_{x=L}=u_{3k-1}(x,t)\big|_{x=0}=u_{3k}(x,t)\big|_{x=0},$$
$$u_{3k-2,x}(x,t)\big|_{x=L}=u_{3k-1,x}(x,t)\big|_{x=0}+u_{3k,x}(x,t)\big|_{x=0}, \tag{4}$$

$$u_{3k}(x,t)\big|_{x=L}=u_{3k-1}(x,t)\big|_{x=L}=u_{3k-2}(x,t)\big|_{x=0},$$
$$u_{3k,x}(x,t)\big|_{x=L}+u_{3k-1,x}(x,t)\big|_{x=L}=u_{3k-2,x}(x,t)\big|_{x=0},\ k=\overline{1,n},\ t\geq 0 \tag{5}$$

satisfactory $u_k=(u_1,u_2,u_3,\dots,u_{3n-2})^T$ Let's find a solution.

Here $\varphi_k(x)$ and $\psi_k(x)$, $k=\overline{1,3n+2}$ Given functions, sufficiently smooth functions. In addition, we assume that they have the following properties consistent with conditions (3)-(5):

$$\varphi_k(0)=0,\ \psi_k(0)=0,\quad k\in\{1,2\},$$
$$\varphi_k(L)=0,\ \psi_k(L)=0,\quad k\in\{3n+1,3n+2\}, \tag{6}$$

$$\varphi_{3k-2}(L)=\varphi_{3k-1}(0)=\varphi_{3k}(0),$$
$$\varphi'_{3k-2}(L)=\varphi'_{3k-1}(0)+\varphi'_{3k}(0),$$
$$\varphi_{3k}(L)=\varphi_{3k-1}(L)=\varphi_{3k-2}(0),$$
$$\varphi'_{3k}(L)+\varphi'_{3k-1}(L)=\varphi'_{3k-2}(0),\quad k=\overline{1,n}, \tag{7}$$

$$\psi_{3k-2}(L)=\psi_{3k-1}(0)=\psi_{3k}(0),$$
$$\psi'_{3k-2}(L)=\psi'_{3k-1}(0)+\psi'_{3k}(0),$$
$$\psi_{3k}(L)=\psi_{3k-1}(L)=\psi_{3k-2}(0),$$
$$\psi'_{3k}(L)+\psi'_{3k-1}(L)=\psi'_{3k-2}(0),\quad k=\overline{1,n}. \tag{8}$$

2. Spectral problem in metric graphs

We solve the homogeneous part of equations (1) using the separation of variables method (Fourier method)

$$X''_k(x)+\lambda^2X_k(x)=0,\ k=\overline{1,3n+2}, \tag{9}$$

If we satisfy conditions (3)-(5),

$$X_j(0)=0,\quad j\in\{1,2\},\quad X_l(L)=0,\quad l\in\{3n+1,3n+2\} \tag{10}$$

$$X_{3k-2}(L)=X_{3k-1}(0)=X_{3k}(0), \tag{11}$$

$$X'_{3k-2}(L)-X'_{3k-1}(0)-X'_{3k}(0)=0, \tag{12}$$

$$X_{3k}(L)=X_{3k-1}(L)=X_{3k-2}(0), \tag{13}$$

$$X'_{3k}(L)+X'_{3k-1}(L)-X'_{3k-2}(0)=0,\ k=\overline{1,n} \tag{14}$$

we will have relationships.

Problems of the form (9)-(14) are usually called Sturm-Liouville spectral problems. Spectral problems on metric graphs can be found in the literature [5]-[6].

It is known from the course of differential equations that the general solution of equation (9) is as follows

$$X_k(x) = c_k \cos \lambda x + d_k \sin \lambda x \tag{15}$$

has the form. (15) on each side of the equation c_k and d_k Using conditions (10)-(14) with respect to the unknown coefficients,

$$M(\lambda)Q = 0 \tag{16}$$

we get a system of equations. Here Q is a vector composed of unknown coefficients and is written in the following form:

$$Q = (c_1, d_1, c_2, d_2, \ldots, c_{3n-2}, d_{3n-2})^T.$$

Considering conditions (10)-(14), we have $M(\lambda)$ the matrix will look like this:

$$M_{6n-2\,6n-2} = \begin{pmatrix} A_{6\,0} & B_{6\,4} & C_{6\,4} & A_{6\,6(n-1)} \\ A_{6\,4} & D_{6\,6} & C_{6\,4} & A_{6\,6(n-2)} \\ A_{6\,10} & D_{6\,6} & C_{6\,4} & A_{6\,6(n-3)} \\ \ldots & \ldots & \ldots & \ldots \\ A_{6\,4+(n-2)6} & D_{6\,6} & C_{6\,4} & A_{6\,0} \\ 0 & 0 & 0 & E_{2\,4} \end{pmatrix}$$

We can say that, $M_{6n-2\,6n-2}(\lambda)$ matrix $B_{6\,4}$, $D_{6\,6}$ and $E_{2\,4}$ is a diagonal matrix with respect to matrices. Here, A_{ij} matrices are zero matrices. For convenience, if $\cos \lambda L$ and $\sin \lambda L$ for values $a = \cos \lambda L$, $b = \sin \lambda L$ If we enter the definitions, $B_{6\,4}$, $C_{6\,4}$, $D_{6\,6}$ and $E_{2\,4}$ Matrices are expressed in the following forms:

$$B_{64}=\begin{pmatrix} b & 0 & -1 & 0 \\ b & 0 & 0 & 0 \\ -a & 0 & 0 & 1 \\ 0 & -b & a & b \\ 0 & 0 & a & b \\ 0 & -a & b & -a \end{pmatrix}, \qquad C_{64}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$D_{66}=\begin{pmatrix} a & b & 0 & 0 & -1 & 0 \\ a & b & 0 & 0 & 0 & 0 \\ b & -a & 0 & 0 & 0 & 1 \\ 0 & 0 & a & b & -a & -b \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & b & -a & b & -a \end{pmatrix}, \qquad E_{24}=\begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & a & b \end{pmatrix}.$$

We are below $n=1$ Let's give an example for the case where . That is, the number of edges 5 For the graph with ta, we consider. According to equality (15) and conditions (3) - (5),

$$\sin^3 \lambda L\left(8\cos^2 \lambda L-\sin^2 \lambda L\right)=0, \quad \lambda_{1,m}=\lambda_{2,m}=\lambda_{3,m}=\frac{\pi m}{L}, \ m\in N$$

$$\lambda_{4,m}=\lambda_{5,m}=\pm\frac{1}{2L}\arccos\left(-\frac{7}{9}\right)+\frac{\pi m}{L}, \ m\in N$$

in appearance $\lambda_{k,m}$ ($k=\overline{1,5}$) we have the eigenvalues. Taking into account equality (15), we find the corresponding eigenvalues $X_{km}(x)$ ($k=\overline{1,5}$) we write the eigenfunctions

$$X_{1m}(x)=c_1\cos\frac{\pi m}{L}x+d_1\sin\frac{\pi m}{L}x, \quad X_{2m}(x)=c_2\cos\frac{\pi m}{L}x+d_2\sin\frac{\pi m}{L}x,$$

$$X_{3m}(x)=c_3\cos\frac{\pi m}{L}x+d_3\sin\frac{\pi m}{L}x, \quad X_{4m}(x)=c_4\cos\lambda_{4,m}x+d_4\sin\lambda_{4,m}x,$$

$$X_{5m}(x)=c_5\cos\lambda_{5,m}x+d_5\sin\lambda_{5,m}x.$$

3. Problem solution research: existence and uniqueness of the solution

Theorem 1. If $\varphi_k(x), \psi_k(x) \hat{I} C^1[0,L]$, $\frac{\P}{\P x} f_k(x,t) \hat{I} C\left([0,L] \times (0,T)\right)$ $k=\overline{1,3n+2}$ are functions defined in classes, $[0,L]$ at intervals $\varphi_k''(x)$, $\psi_k''(x)$ and

$\left|(0,L)\times(0,T)\right|$ in the field $\frac{\partial^2}{\partial x^2} f_k(x,t)$ if the functions are absolutely integrable, then $\varphi_k(x)$ for functions

$$\varphi_k''(0)=0,\ k\in\{1,2\},\ \varphi_k''(L)=0,\ k\in\{3n+1,3n+2\},$$

$$\varphi_{3k}''(L,t)=\varphi_{3k-2}''(0,t)=\varphi_{3k-1}''(0,t),\ k=\overline{1,n},$$

$$\varphi_{3k}''(L,t)=\varphi_{3k-1}''(L,t)=\varphi_{3k-2}''(0,t),\ k=\overline{1,n}$$

conditions are appropriate and conditions (3)-(5) and (6)-(8) are satisfied respectively $\psi_k(x)$, $f_k(x,t)$, $k=\overline{1,3n+2}$ is also valid for functions, then the solution to problem (1)–(5) exists and is unique.

PROOF. Given. $f(x,t)$ We expand the functions into a Fourier series using the eigenfunctions.

$$f(x,t)=\sum_{m=0}^{\infty} f_m(t)X_m(x) \tag{17}$$

Here $f_m(t)$ (17) The coefficients of the Fourier series. If we write the solution of equation (1) in the following form,

$$u(x,t)=\sum_{m=0}^{\infty} X_m(x)W_m(t),\ k=\overline{1,3n+2} \tag{18}$$

Using this solution and equation (1),

$$\sum_{m=0}^{\infty}(W_m''(t)+a^2\lambda_m^2W_m(t)-f_m(t))X_m(x)=0 \tag{19}$$

we form an equation. $X_m(x)$ Since the function is a specific function of the problem under consideration, this

$$W_m''(t)+a^2\lambda_m^2W_m(t)=f_m(t) \tag{20}$$

we have a non-homogeneous differential equation. It is known that the general solution of equation (20) is

$$W_m(t)=\frac{1}{a\lambda_m}\int_0^t \sin a\lambda_m(t-z)f_m(z)dz+c_{1m}\cos a\lambda_m t+c_{2m}\sin a\lambda_m t \tag{21}$$

[9] Therefore, based on (18), we can write the general solution of equation (1) as follows:

$$u_k(x,t)=\sum_{m=0}^{\infty}\left[\frac{1}{a\lambda_m}\int_0^t \sin a\lambda_m(t-z)f_m(z)dz+c_{1m}\cos a\lambda_m t+c_{2m}\sin a\lambda_m t\right]X_{km}(x)$$

$k=\overline{1,3n+2}$.(22)

It is known that given $\varphi_k(x)$,$\psi_k(x)$ functions as follows

$$\varphi_k(x)=\sum_{m=0}^{\infty}\varphi_m X_{km}(x),\quad \psi_k(x)=\sum_{m=0}^{\infty}\psi_m X_{km}(x) \tag{23}$$

We can expand it into Fourier series in the form of . Here φ_m and ψ_m (23) are the coefficients of the Fourier series. If we satisfy the initial condition (2) for the obtained general solution of (22),

$$c_{1m}=\varphi_m,\quad c_{2m}=\frac{\psi_m}{a\lambda_m} \tag{24}$$

we will have equalities.

Therefore, based on equations (22) and (24), the solution of equation (1) satisfying conditions (2) is found as follows:

$$u_k(x,t)=$$

$$=\sum_{m=0}^{\infty}\left[\frac{1}{a\lambda_m}\int_0^t \sin a\lambda_m(t-z)f_m(z)dz+\varphi_m\cos a\lambda_m t+\frac{\psi_m}{a\lambda_m}\sin a\lambda_m t\right]X_{km}(x). \tag{25}$$

Given the equations (23) $\varphi_k(x)$,$\psi_k(x)$ functions $\varphi_k(x)$,$\psi_k(x)\in L_2[0,L]$ since

$$\varphi_m=\sum_k\int_0^L\varphi_k(x)X_{km}(x)dx,\quad \psi_m=\sum_k\int_0^L\psi_k(x)X_{km}(x)dx \tag{26}$$

We can write it in the form: (26) Integrating the right-hand side of the equations by parts two and three times, respectively, and taking into account the conditions (6)-(8) and the conditions of the theorem,

$$\varphi_m=\frac{1}{\lambda_m^3}\sum_{k=1}^{3n-2}\int_0^L\varphi_k'''(x)X_{km}^*(x)dx,\quad \psi_m=-\frac{1}{\lambda_m^2}\sum_{k=1}^{3n-2}\int_0^L\psi_k''(x)X_{km}(x)dx. \tag{27}$$

we get the equalities, here $X_{km}^*(x)=c_k\sin\lambda_m x-d_k\cos\lambda_m x$.

Like, $\varphi_k(x)$,$\psi_k(x)$ similar to the functions, given in equation (17) $f_k(x,t)$ function too $L_2[0,L]$ from belonging to the class, from the conditions of the theorem

$$f_m(t)=\sum_k\int_0^L f_k(x,t)X_{km}(x)dx=-\frac{1}{\lambda_m^2}\sum_{k=1}^{3n-2}\int_0^L\frac{d^2}{dx^2}f_k(x,t)X_{km}(x)dx \tag{28}$$

we will have equality.

It is known that in order for the function found in the form (25) to be a solution to the given problem, this function must also satisfy equation (1). Therefore, in the area under consideration $u_k(x,t)$, $u_{k\,xx}(x,t)$, $u_{k\,tt}(x,t)$ It is required to prove that the functions are defined.

Lemma 1. $X_{km}(x)=c_k\sin\lambda_m x+d_k\cos\lambda_m x$ The following inequality holds for the eigenfunction.

$$|X_{km}(x)|=|c_{km}\cos\lambda_m x+d_{km}\sin\lambda_m x|\le\sqrt{\frac{2}{L}}\,. \tag{29}$$

Therefore, we can write the solution to (25) in the following form.

$$u_k(x,t)=\sum_{m=0}^{\infty}\left[\frac{M_1}{a\lambda_m^3}\int_0^t\sin a\lambda_m(t-z)dz+\frac{M_2}{\lambda_m^3}\cos a\lambda_m t+\frac{M_3}{a\lambda_m^3}\sin a\lambda_m t\right]\times$$
$$\times\sqrt{c_{km}^2+d_{km}^2}\sin(\lambda_m x+A_{km})$$

Here M_1, M_2, M_3 -immutable numbers and $\cos A_{km}=\frac{d_{km}}{\sqrt{c_{km}^2+d_{km}^2}}$, $\sin A_{km}=\frac{c_{km}}{\sqrt{c_{km}^2+d_{km}^2}}$.

Lemma 2 If the following

$$\max\left|\frac{M_1}{a}\sqrt{c_{km}^2+d_{km}^2}\cdot\int_0^t\sin a\lambda_m(t-z)dz\right|=M_{12}<+\infty,$$

$$\max\left|M_2\sqrt{c_{km}^2+d_{km}^2}\cdot\cos a\lambda_m t\right|=M_{22}<+\infty,$$

$$\max\left|\frac{M_3}{a}\sqrt{c_{km}^2+d_{km}^2}\cdot\sin a\lambda_m t\right|=M_{32}<+\infty,$$

If the conditions are correct, line (25) and its x and t generated by taking the derivative twice with respect to the variables $u_{k\,xx}(x,t)$, $u_{k\,tt}(x,t)$ The series also converge smoothly in the domain under consideration. Here M_{12}, M_{22}, M_{32} - immutable numbers.

LemWe know the proof of ma 2 [8]. From this we have proved that all infinite series and their derivatives corresponding to the solution converge. This means that the problem posed exists.

Lemma 3. The issue under consideration $u_k(x,t)$ The following inequality holds for the solution.

$$\int_0^t \|u_\tau(x,\tau)\|_\Gamma^2 d\tau \le h_1 \int_0^t (t-\tau)\|f(x,\tau)\|_\Gamma^2 d\tau + h_2\|\psi(x)\|_\Gamma^2 + h_3\|\varphi_x(x)\|_\Gamma^2$$

Here h_i , $i=1,2,3$ are immutable numbers.

PROOF. Equation (1) $u_{kt}(x,t)$ We multiply the scalar by , Γ By definition, the scalar product in a graph is

$$\left(u_{kt}(x,t),u_{tt}(x,t)\right)_\Gamma - \left(u_{kt}(x,t),a^2u_{xx}(x,t)\right)_\Gamma = \left(u_{kt}(x,t),f(x,t)\right)_\Gamma,$$

$$\sum_k \int_0^L u_{kt}(x,t)u_{ktt}(x,t)dx - a^2\sum_k \int_0^L u_{kt}(x,t)u_{kxx}(x,t)dx =$$

$$= \sum_k \int_0^L u_{kt}(x,t)f_k(x,t)dx. \tag{30}$$

Equality (30) takes the following form according to the conditions of the problem:

$$\frac{1}{2}\frac{d}{dt}\|u_t(x,t)\|_\Gamma^2 + \frac{a^2}{2}\frac{d}{dt}\|u_x(x,t)\|_\Gamma^2 \le \frac{1}{2}\|u_t(x,t)\|_\Gamma^2 + \frac{1}{2}\|f(x,t)\|_\Gamma^2. \tag{31}$$

Now the result obtained 0 from t in between τ If we integrate over,

$$\|u_t(x,t)\|_\Gamma^2 + a^2\|u_x(x,t)\|_\Gamma^2 \le \int_0^t \left(\|u_\tau(x,\tau)\|_\Gamma^2 + \|f(x,\tau)\|_\Gamma^2\right)d\tau +$$

$$+\|u_t(x,0)\|_\Gamma^2 + a^2\|u_x(x,0)\|_\Gamma^2.$$

we get the inequality in the form: (2) Taking into account condition,

$$\|u_t(x,t)\|_\Gamma^2 + a^2\|u_x(x,t)\|_\Gamma^2 \le$$

$$\le \int_0^t \left(\|u_\tau(x,\tau)\|_\Gamma^2 + \|f(x,\tau)\|_\Gamma^2\right)d\tau + \|\psi(x)\|_\Gamma^2 + a^2\|\varphi_x(x)\|_\Gamma^2. \tag{32}$$

We know that inequality (32) is equally strong as the following inequalities:

$$a^2\|u_x(x,t)\|_\Gamma^2 \le \int_0^t \|u_\tau(x,\tau)\|_\Gamma^2 d\tau + \int_0^t \|f(x,\tau)\|_\Gamma^2 d\tau + c_1\|\psi(x)\|_\Gamma^2 + c_2\|\varphi_x(x)\|_\Gamma^2,$$

$$\|u_t(x,t)\|_\Gamma^2 \le \int_0^t \|u_\tau(x,\tau)\|_\Gamma^2 d\tau + \int_0^t \|f(x,\tau)\|_\Gamma^2 d\tau + c_1\|\psi(x)\|_\Gamma^2 + c_2\|\varphi_x(x)\|_\Gamma^2. \tag{33}$$

In inequality (33) $y(t)=\int_0^t \|u_\tau(x,\tau)\|_\Gamma^2 d\tau$ that, using the Gronwall-Bellman inequality[7], $y(0)=0$ since

$$\int_0^t \|u_\tau(x,\tau)\|_\Gamma^2 d\tau \le e^t\left[\int_0^t ds\int_0^s \|f(x,\tau)\|_\Gamma^2 d\tau+\int_0^t \|\psi(x)\|_\Gamma^2 d\tau+\int_0^t \|\varphi_x(x)\|_\Gamma^2 d\tau\right]=$$

$$=h_1\int_0^t (t-\tau)\|f(x,\tau)\|_\Gamma^2 d\tau+h_2\|\psi(x)\|_\Gamma^2+h_3\|\varphi_x(x)\|_\Gamma^2 \quad (34)$$

we create an inequality. h_1, h_2 and h_3 are positive invariants. Lemma 3 has been proved.

Using standard methods, it is possible to prove that the solution to problems (1)-(5) is unique. If, in inequality (34), $f_k(x,\tau)\equiv 0$, $\psi_k(x)\equiv 0$ and $\varphi_k(x)\equiv 0$ If we assume that, then we have, $u_k(x,t)\equiv 0$.

Therefore, the solution to problem (1)-(5) exists and is unique. Theorem 1 is completely proven.

REFERENCES USED

1. K. Nakamura, ZA Sobirov, DU Matrasulov and S. Sawada, Transport in simple networks described by an integrable discrete nonlinear Schrodinger equation, Phys. Rev E 84, 026609 (2011)
2. ZA Sobirov, MI Akhmedov, H. Uecker, Cauchy problem for the linearized KdV equation on general metric star graphs, Nanosystems: Physics, Chemistry, Mathematics, 2015, 6(1). P. 198-204.
3. G. Berkolaiko, An elementary introduction to quantum graphs, archive: 1603.07356v2 [math-ph] 17 dec. 2016.
4. MR Eshimbetov, Initial-boundary value problem for heat equation on ladder-type graph, Bulletin of the Institute of Mathematics 2020, №??, pp.1-8
5. G. Berkolaiko and P. Kuchment, Dependence of the spectrum of a quantum graph on vertex conditions and edge lengths., In Spectral Geometry, volume 84 of Proceedings of Symposia in Pure Mathematics. American Math. Soc., Providence 117-137. 2012.
6. S. Gnutzmann, JP Keating, F. Piotet, Quantum ergodicity on graphs, Physical Review Letters. PRL 101, 264102 (2008) DOI: 10.1103/PhysRevLett.101.264102

7. OA Ladyzhenskaya, Kraevye zadachi matematicheskoi fizik (Boundary Value Problems of mathematical Physics), Moscow: Nauka. 1973. (in Russian).

8. L.D. Kudryavtsev, Course of mathematical analysis, Volume 2. 2004, 720p

9. N. Dilmurodov, Ordinary Differential Equations, Tashkent: “Sano-Standard” Publishing House, 2019. 476 pages

10. MH Kenjaboyeva, Theorem on the uniqueness of the solution of the initial-boundary problem for the wave propagation equation in staircase graphs, Scientific Conference on “Differential Equations and Related Problems of Analysis”, Bukhara, Uzbekistan, November 4-5, 2021. Page 165.

11. JR Khujakulov, MH Kenjaboyeva, Initial-boundary problem for the wave propagation equation in staircase graphs, International Scientific-Practical Conference on "Modern Problems of Mathematical Physics and Mathematical Modeling", Karshi, Uzbekistan, December 3-4, 2021. Page 341.

12. Fayazova, D. (2024). SKILLS TO ENHANCE THE EFFECTIVENESS OF MATHEMATICS LESSONS IN PRIMARY EDUCATION. Western European Journal of Modern Experiments and Scientific Methods, 2(10), 68-76.

13. Фаязова, Д. Т. (2023). Повышение мотивированности студентов направления химии к изучению высшей математики. SCHOLAR, 1(13), 172-176.

14. Фаязова, Д. Т. (2023). Условия формирования профессиональных компетенций. Scientific Impulse, 1(8), 565-570.