PROPERTIES OF HOM AND TENZOR PRODUCT

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Abstract: The article presents the notion of Hom and tenzor product as well as several intriguing properties of them that are related to □ , □$_m$ and □ .

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Firstly, let's recall the notion of tenzor product. Let $A \in Mod_R, B \in {}_R Mod \Rightarrow$ we will construct $A \underset{R}{\otimes} B$ - an abelian group. Let $C \in Ab$.

Definition: A middle linear map $f : A \times B \to C$ is defined by

(1) $f(a_1 \underset{A}{+} a_2, b) = f(a_1, b) \underset{C}{+} f(a_2, b)$ and (2) $f(a, b_1 \underset{B}{+} b_2) = f(a, b_1) \underset{C}{+} f(a, b_2)$

(3) $f(a.r, b) = f(a, r.b)$

Fix A and B as above. Let's define category $\mathrm{M}(A, B)$:

1. Objects are middle linear maps on $A \times B$, i.e $\{f : A \times B \to \_\}$
2. Morphisms between $f : A \times B \to C$ and $g : A \times B \to D$ is a group homomorphism $h : C \to D$ s.t the diagram is commutative: $g = h \circ f$

An initial object in $\mathrm{M}(A, B)$, if it exists, has the universal property: $\forall (f, C) \in Ob(M(A, B))$ there exists $\alpha : init.ob \to (f, C)$ s.t the diagram commutes. This means that if we denote the initial object by $(i, A \otimes B)$ then we have

$A \times B \xrightarrow{i} A \otimes B \xrightarrow{\alpha} C$ and $f = \alpha \circ i$. $A \in Mod_R, B \in {}_R Mod$, F be a free abelian group on the set $A \times B$. That is $A \times B \xrightarrow{i} F$ and $A \times B \xrightarrow{\forall f} \forall C \in Ab$, then $\exists h : F \to C$ an abelian group homomorphism s.t $f = h \circ i$.

The elements of F are $(a, b), (a_1, b_1) + (a_2, b_2), \ldots$

Definition: Let K be a subgroup of F generated by all elements of the following form: (1) $(a+a',b)-(a,b)-(a',b)$ (2) $(a,b+b')-(a,b)-(a,b')$ (3) $(a.r,b)-(a,r.b)$

The quotient group F/K is called the tensor product of A and B and is denoted $A\underset{R}{\otimes}B$. The coset $(a,b)+K$ of the element $(a,b)\in F$ is denoted by $a\otimes b$. Note that in $A\otimes B$ we have

(1) $(a+a')\otimes b = a\otimes b + a'\otimes b$ (2) $a\otimes(b+b') = a\otimes b + a\otimes b'$

(3) $a.r\otimes b = a\otimes r.b$ (*)

Note that $(a,b)\overset{i}{\mapsto}(a,b)+K = a\otimes b$ is a middle linear map, this is because of (*). We call this $i: A\times B\to A\underset{R}{\otimes}B$ the canonical middle linear map. Now $(i, A\underset{R}{\otimes}B)$ will be the initial object in $\mathrm{M}(A,B)$

Property : (a) $\forall m\in\mathbb{N}, m>0, \forall A$ is abelian group, prove that $Hom(\mathbb{Z}_m, A)\cong A[m] := \{a\in A \mid ma=0\}$

(b) $Hom(\mathbb{Z}_m,\mathbb{Z}_n)\cong\mathbb{Z}_{\gcd(m,n)}$ (c) $\mathbb{Z}_m^{*} = Hom_{\mathbb{Z}}(\mathbb{Z}_m,\mathbb{Q}) = 0$

Proof: (a) $\mathbb{Z}_m = \{\bar{1},\bar{2},....(\overline{m-1})\}$, $f:\mathbb{Z}_m\to A$ - homomorphism of abelian groups. Let $f(1)=a \Rightarrow f(\underbrace{\bar{1}+\bar{1}+...+\bar{1}}_{m}) = \underbrace{f(\bar{1})+f(\bar{1})+...+f(\bar{1})}_{m} = ma$ and we know that

$\underbrace{\bar{1}+\bar{1}+...+\bar{1}}_{m} = 0 \Rightarrow 0 = f(\bar{0}) = f(m\bar{1}) = ma$, so we can define $\varphi: Hom(\mathbb{Z}_m, A)\to A[m]$

s.t $f\mapsto f(\bar{1})$, it is easy to check being homomorphism:

$f\mapsto f(\bar{1})\, f+g\mapsto (f+g)(\bar{1}) = f(\bar{1})+g(\bar{1})$.

Now we define $\psi: A[m]\to Hom(\mathbb{Z}_m, A)$ s.t $a\mapsto f$, $f(\bar{1})=a$ ($\mathbb{Z}_m$ has generator $\bar{1}$, so it is enough to define f at $\bar{1}$), $ma=0\Rightarrow f\in Hom(\mathbb{Z}_m, A)$.

Let $\bar{k}=\bar{k'} \Rightarrow f(\bar{k}) = f(\underbrace{\bar{1}+\bar{1}+...+\bar{1}}_{k}) = ka = (k'+mn)a = k'a = f(k') \Rightarrow$ well-defined.

$f(\bar{k}+\bar{n}) = (k+n)a = ka+na = f(\bar{k})+f(\bar{n})$ and $a+a' \mapsto f, f(\bar{1}) = a+a'$

$a \mapsto g, g(\bar{1}) = a$ and $a' \mapsto h, h(\bar{1}) = a'$, therefore

$(g+h)(\bar{1}) = g(\bar{1})+h(\bar{1}) = a+a' = f(\bar{1})$. So $g+h$ and f are equal at generator , thus they are exactly the same functions.

Now, we consider the composition of these homomorphisms, if they give identity, then $Hom(\mathbb{Z}_m, A) \cong A[m]$: $(\varphi \circ \psi)(a) = \varphi(\psi(a)) = \psi(a)(\bar{1}) = a$ and

$(\psi \circ \varphi)(f) = \psi(f(\bar{1})) = f$

(b) $Hom(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{\gcd(m,n)}$

From (a) we have $Hom(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_n[m] = \{\bar{k} \in \mathbb{Z}_n \mid m\bar{k} = \bar{0}\}$, so it is enough to show that $\mathbb{Z}_n[m] \cong \mathbb{Z}_{\gcd(m,n)}$. Let $m = m'd, n = n'd, (m',n') = 1$, we have to find $\bar{k} \in \mathbb{Z}_n, m\bar{k} \vdots n$ (we can assume that $0 \le k \le n-1$). In order to accomplish this $mk \vdots n$ must be: $m = m'dk \vdots n = n'd \Leftrightarrow m'k \vdots n' \Leftrightarrow k \vdots n' \Rightarrow k = \{0, n', ..., (d-1)n'\}$

Now we can construct $\varphi: \mathbb{Z}_n[m] \to \mathbb{Z}_d$ s.t $\{0, n', ..., (d-1)n'\} \to \mathbb{Z}_d = \{\bar{0}, \bar{1}, ..., \overline{d-1}\}$

It is making a sense that $\overline{kn'} \mapsto \bar{k}$ and by checking $\overline{kn'} + \overline{sn'} = \overline{(s+k)n'} \mapsto \overline{s+k} = \bar{s} + \bar{k}$

It is easy to see that φ is bijective $\mathbb{Z}_n[m] \cong \mathbb{Z}_{\gcd(m,n)}$.

(c) $\mathbb{Z}_m^* = Hom_{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}) = 0$

According to (a) we have $\mathbb{Z}_m^* \cong \mathbb{Z}[m] = \{k \in \mathbb{Z}, mk = 0 \Leftrightarrow k = 0\} = 0$.

Property : Let A is abelian group. Prove that:

(a) $\forall m \in \mathbb{Z}, m > 0, A \otimes \mathbb{Z}_m \cong A/mA$

$(b)\ \mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_{\gcd(m,n)}$ $(c)\ \square \otimes \square \cong \square$

Proof: (a) Initially we construct $h^0 : A \times \mathbb{Z}_m \to A/mA$ s.t $(a,\bar{k}) \mapsto ka + mA$

$\bar{k} = \bar{k}' \Rightarrow k = k' + mt$ and

$(a,\bar{k}') \mapsto k'a + mA = (k - mt)a + mA = ka - tma + mA = ka + mA$. Because of $(a,\bar{k}) \mapsto ka + mA$ we get $(a,\bar{k}') = (a,\bar{k})$. So the function is well-defined. Now we examine that it is middle linear map:

$1)(a + a', \bar{k}) \mapsto k(a + a') + mA = (ka + mA) + (ka' + mA)$

$2)(a, \bar{k} + \bar{n}) = (a, \overline{k + n}) \mapsto (k + n)a + mA = (ka + mA) + (na + mA)$

$3)(na, \bar{k}) \mapsto k(na) + mA = (kn)a + mA$

So, by universal property we get $h : A \otimes \mathbb{Z}_m \to A/m$ which is homomorphism of groups s.t $a \otimes \bar{k} \mapsto ka + mA$

Now we define $g : A/mA \to A \otimes \mathbb{Z}_m$ s.t $a + mA \mapsto a \otimes \bar{1}$. For simplicity lets define $a + mA = [a]$, then $[a] + [b] \mapsto (a + b) \otimes \bar{1} = a \otimes \bar{1} + b \otimes \bar{1}$, thus g is homomorphism of groups. $[a] = [a'] \Rightarrow a - a' = ma'', [a] = a \otimes \bar{1} = (a' + ma'') \otimes \bar{1} =$

$= a' \otimes \bar{1} + ma'' \otimes \bar{1} = a' \otimes \bar{1} + a'' \otimes m\bar{1} = a' \otimes \bar{1} + a'' \otimes \bar{0} = a' \otimes \bar{1}$, thus g is well-defined. $(g \circ h)(a \otimes \bar{k}) = g([ka]) = ka \otimes \bar{1} = a \otimes \bar{k}$ and $(h \circ g)([a]) = h(a \otimes \bar{1}) = [a]$ $\Rightarrow g \circ h = id_{A \otimes \mathbb{Z}_m}$ at generators of $A \otimes \mathbb{Z}_m$, so they are equal at full group.

$g \circ h = id_{A/mA} \Rightarrow A \otimes \mathbb{Z}_m \cong A/mA$

(b) Denote $\bar{k}_m$ as class $\bar{k}$ in $\mathbb{Z}_m$. $h_0 : \mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_d$ s.t $(\overline{k_m}, \overline{s_n}) \mapsto \overline{ks_d}$. Now we check whether it is well-defined:

$\overline{k_m} = \overline{k'_m}, \overline{s_n} = \overline{s'_n} \Rightarrow ks = (k' + mt)(s' + nl) = k's' + k'nl + s'mt + mnlt \equiv k's' (\bmod d)$

$\Rightarrow \overline{ks_d} = \overline{ks'_d}$. So it is well-defined and now we show it is a middle linear map:

$1)(\overline{k_m}+\overline{k'_m},\overline{s_n})\mapsto\overline{(k+k')s_d}=\overline{ks_d}+\overline{k's_d}$

$2)(\overline{k_m},\overline{s_n}+\overline{s'_n})=(\overline{k_m},\overline{s_n+s'_n})\mapsto\overline{k(s+s')_d}=\overline{ks_d}+\overline{ks'_d}$

$3)(l\overline{k_m},\overline{s_n})=(\overline{lk_m},\overline{s_n})\mapsto\overline{lks_d}=\overline{kls_d}\Rightarrow\exists h:\mathbb{Z}_m\otimes\mathbb{Z}_n\to\mathbb{Z}_d$ s.t $h(\overline{k},\overline{s})=\overline{ks}$ which is group homomorphism. Let $g:\mathbb{Z}_d\to\mathbb{Z}_m\otimes\mathbb{Z}_n$ s.t $\overline{s_d}\mapsto\overline{s_m}\otimes\overline{1_n}$

And their compositions give identity maps: $(h\circ g)(\overline{s_d})=h(\overline{s_d}\otimes\overline{1_n})=\overline{s_d}$ and

$(g\circ h)(\overline{k_m}\otimes\overline{s_n})=g(\overline{ks_d})=\overline{ks_m}\otimes\overline{1_n}=s\overline{k_m}\otimes\overline{1_n}=\overline{k_m}\otimes\overline{s_n}$

$(c)\ \mathbb{Q}\times\mathbb{Q}\to\mathbb{Q},(r,s)\mapsto rs$ this is a middle linear map, because:

$1)(r+r',s)\mapsto(r+r')s=rs+r's$ and $2)(r,s+s')\mapsto r(s+s')=rs+rs'$,

$3)(nr,s)=nrs=r(ns)\Rightarrow\exists\varphi:\mathbb{Q}\otimes\mathbb{Q}\to\mathbb{Q},r\otimes s\mapsto rs$. Now $\psi:\mathbb{Q}\to\mathbb{Q}\times\mathbb{Q},r\mapsto r\otimes 1$, it is a homomorphism : $(r+s)\mapsto(r+s)\otimes 1=r\otimes 1+s\otimes 1$

Now we will examine their compositions:

$$r\otimes s\mapsto rs\mapsto rs\otimes 1=\frac{p}{q}\cdot\frac{l}{k}\otimes 1=\frac{p}{q}\cdot\frac{l}{k}\otimes k\cdot\frac{1}{k}=\frac{p}{q}\cdot\frac{l}{k}\cdot k\otimes\frac{1}{k}=\frac{pl}{q}\otimes\frac{1}{k}=\frac{p}{q}\otimes\frac{l}{k}=r\otimes s$$

$r\mapsto r\otimes 1\mapsto r$, consequently, $\varphi\circ\psi=id_{\mathbb{Q}}$ and $\psi\circ\varphi=id_{\mathbb{Q}\otimes\mathbb{Q}}$.

References:

1. Thomas W. Hungerford, Algebra. 2000, USA.
2. Evan Chen, An Infinitely Large Napkin. August 31, 2020
3. ttps://artofproblemsolving.com/